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Informatique théorique/*Computer Science*
(Théorie des signaux/*Theory of Signals*)

On *P*-simple points

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Abstract – A simple point of an object is a point whose removal does not change the topology. However, the simultaneous deletion of simple points may change the topology. A popular way for overcoming this problem is to use a directional strategy. This method has good properties in two-dimensional discrete spaces but it does not work in three-dimensions. Through the notion of *P*-simple point we propose a general strategy for removing simple points in parallel without altering the topology of a 3D space. We also propose a characterization of *P*-simple points which may be implemented with a linear time complexity.

Sur les points *P*-simples

Résumé – *Un point simple est un point d'un objet dont la supression ne modifie pas la topologie. Cependant la suppression d'un ensemble de points simples peut changer la topologie. Une façon classique de résoudre ce problème est d'utiliser une stratégie directionnelle. Cette méthode donne de bons résultats dans des espaces discrets à deux dimensions mais elle ne convient pas pour des espaces à trois dimensions. À travers la notion de point P-simple nous proposons une stratégie générale pour retirer en parallèle des points simples sans modifier la topologie d'un espace 3D. Nous proposons également une caractérisation des points P-simples pouvant être implantée avec un temps de calcul linéaire.*

Version française abrégée – 1. NOTIONS DE BASE. – Un point $x \in \mathbf{Z}^3$ est défini par (x_1, x_2, x_3) avec $x_i \in \mathbf{Z}$. Nous considérons les voisinages :

$$N_{26}(x) = \{x' \in \mathbf{Z}^3;\ \mathrm{Max}\,[|x_1 - x_1'|,\ |x_2 - x_2'|,\ |x_3 - x_3'|] \leqq 1\},$$
$$N_{26}^*(x) = N_{26}(x)\backslash\{x\},$$
$$N_6(x) = \{x' \in \mathbf{Z}^3;\ |x_1 - x_1'| + |x_2 - x_2'| + |x_3 - x_3'| \leqq 1\},$$
$$N_6^*(x) = N_6(x)\backslash\{x\}.$$

Deux points x et y sont *n-adjacents* si $y \in N_n^*(x)$ avec $n = 6$ ou 26.

Un *n-chemin* est une séquence de points $x_0, ..., x_k$, avec x_i *n*-adjacent à x_{i-1} pour $i = 1...k$. Si $x_0 = x_k$, le chemin est *fermé*.

Soit $X \subset \mathbf{Z}^3$. Deux points $x \in X$ et $y \in X$ sont *n-connectés* s'il existe un *n*-chemin inclus dans X les reliant. Les classes d'équivalence relatives à cette relation sont les *n-composantes connexes* de X. L'ensemble composé des *n*-composantes connexes de X est noté $\mathcal{C}_n(X)$. Afin d'avoir une correspondance entre la topologie de X et celle de $\overline{X}$, il est nécessaire de choisir deux relations d'adjacence différentes pour X et pour $\overline{X}$ (*voir* [4]) : si on utilise une *n*-adjacence pour X, on doit utiliser une $\overline{n}$-adjacence pour $\overline{X}$. Dans cette Note, on considère que $(n, \overline{n}) = (26, 6)$ ou (6, 26).

La présence d'un n-trou dans X est détectée chaque fois qu'il existe un *n*-chemin fermé dans X qui ne peut être déformé, dans X, en un simple point. Soit γ et γ' deux n-chemins fermés. On dit que γ' est une *déformation élémentaire* de γ si γ' et γ sont identiques sauf éventuellement dans une portion P de $\mathbf{Z}^3$; si $n = 26$, P est un cube unité de $2 \times 2 \times 2$ points et si $n = 6$, P est un carré unité de 2×2 points. On dit que γ' est une *déformation* de γ s'il existe une séquence de chemins fermés $\gamma_0, ..., \gamma_k$ telle que $\gamma_0 = \gamma$, $\gamma_k = \gamma'$, γ_i étant une

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déformation élémentaire de γ_{i-1}. La relation de déformation induit des classes d'équivalence qui correspondent à la notion de trou.

Un point $x \in X$ est *n-simple* si sa suppression ne modifie pas la topologie de l'espace en ce sens qu'il existe une bijection entre les composantes, trous de X et $\overline{X}$ et ceux de $X \backslash \{x\}$ et $\overline{X} \cup \{x\}$, la n- et la $\overline{n}$-adjacence étant respectivement utilisées pour X et $\overline{X}$.

2. Points P-simples. – Considérons les nombres topologiques ($\# A$ représente le cardinal de l'ensemble A) :

Définition 1. – Soient $X \subset \mathbf{Z}^3$ et $x \in X$.

Le *n-voisinage géodésique de x dans X à l'ordre k* est l'ensemble $N_n^k(x, X)$ défini récursivement par :

$$N_n^1(x, X) = N_n^*(x) \cap X$$

et

$$N_n^k(x, X) = \bigcup \{N_n(y) \cap N_{26}^*(x) \cap X,\ y \in N_n^{k-1}(x, X)\}.$$

Soient

$$G_6(x, X) = N_6^2(x, X) \qquad \text{et} \qquad G_{26}(x, X) = N_{26}^1(x, X).$$

Les *nombres topologiques* relatifs à X et x sont les nombres :

$$T_n(x, X) = \# \mathcal{C}_n[G_n(x, X)] \qquad \text{avec} \quad n = 6 \quad \text{ou } n = 26.$$

Ces nombres conduisent à une caractérisation des points simples (*voir* [1], [2]) :

Proposition 1. – $x \in X$ *est n-simple si et seulement si* $T_n(x, X) = 1$ *et* $T_{\overline{n}}(x, \overline{X}) = 1$.

En utilisant des algorithmes classiques de recherche de composantes connexes, les calculs de $T_n(x, X)$ et $T_{\overline{n}}(x, \overline{X})$ peuvent être effectués en un temps linéaire; plus précisément ces deux calculs peuvent être faits respectivement en $\mathcal{O}(k)$ et $\mathcal{O}(\overline{k})$, avec $k = \#[G_n(x, X)]$ et $\overline{k} = \#[G_{\overline{n}}(x, \overline{X})]$.

Définition 2. – Soit $X \subset \mathbf{Z}^3$.

Soient $P \subset X$ et $x \in P$: x est *P_n-simple* si $\forall S \subset P \backslash \{x\}$, x est *n*-simple pour $X \backslash S$. On note $S_n(P)$ l'ensemble des points P_n-simples. Un ensemble D est *P_n-simple* si $D \subset S_n(P)$.

Soient $P \subset X$ et $D \subset S_n(P)$. Posons $D = \{x_1, ..., x_k\}$. Supposons que nous enlevions séquentiellement les points $x_1, ..., x_k$. A l'étape *i* de cette procédure, x_i étant P_n-simple, x_i est *n*-simple pour l'ensemble $X \backslash D_i$ avec $D_i = \{x_1, ..., x_{i-1}\}$. Ainsi X et $X \backslash D$ sont topologiquement équivalents au sens décrit ci-dessus : tout algorithme qui n'enlève que des ensembles P_n-simples conserve la topologie.

La caractérisation d'un point simple étant locale, on peut immédiatement proposer une caractérisation également locale des points P-simples :

$$x \text{ est } P_n\text{-simple} \quad \Leftrightarrow \quad \forall S \subset P \cap N_{26}^*(x), \qquad x \text{ est } n\text{-simple pour } X \backslash S.$$

Ainsi, il est possible de vérifier si un point est P_n-simple en testant la condition $[T_n(x, X \backslash S) = 1$ et $T_{\overline{n}}(x, \overline{X \backslash S}) = 1]$ pour tous les ensembles $S \subset P \cap N_{26}^*(x)$. Cette méthode a une complexité de calcul pire que $\mathcal{O}(2^k)$. Elle est donc inutilisable. La proposition suivante permet d'avoir une caractérisation efficace :

Proposition 2. – *Soit* $P \subset X$ *et* $x \in P$; *nous notons* $R = X \backslash P$:

$$x \textit{ est } P_n\textit{-simple} \quad \Leftrightarrow \quad \begin{cases} 1)\ T_n(x, R) = 1 \\ 2)\ T_{\overline{n}}(x, \overline{X}) = 1 \\ 3)\ \forall y \in N_n^*(x) \cap P,\ N_n^*(y) \cap G_n(x, R) \neq \varnothing \\ 4)\ \forall y \in N_{\overline{n}}^*(x) \cap P,\ N_{\overline{n}}^*(y) \cap G_{\overline{n}}(x, \overline{X}) \neq \varnothing. \end{cases}$$

Les conditions 1) et 3) peuvent être implantées en $\mathcal{O}(k)$ et les conditions 2) et 4) en $\mathcal{O}(\overline{k})$: la caractérisation des points *P*-simples peut être faite en un temps linéaire.

3. Un schéma d'amincissement parallèle. – Nous proposons un schéma directionnel général pour amincir des objets 3D. On note $d^{\alpha}(x)$, $\alpha = 0, ..., 5$, les 6-voisins de x [voir *fig.* 2 (*b*)]. Soit $d^{\alpha}(X) = \{x \in X, d^{\alpha}(x) \in \overline{X}\}$. Le schéma consiste à balayer successivement les 6 directions α et à ne retenir comme candidats à la suppression que les points de $d^{\alpha}(X)$. De plus, une notion de point terminal est utilisée afin de garder les surfaces et les courbes qui apparaissent durant l'amincissement de l'objet : l'ensemble de points terminaux d'un ensemble Y est noté $E(Y)$. Le schéma peut alors être décrit de la façon suivante; on pose $X^0 = X$, [ ] représente la fonction modulo :

$$X^{i+1} = X^i \backslash S_n(P^i), \qquad \text{avec} \quad P^i = \overline{E(X^i)} \cap d^{i\,[6]}(X^i).$$

Le *squelette* de X est l'ensemble $\mathrm{SK}(X) = X^k$, k étant tel que $X^k = X^{k+6}$.

Ce schéma conduit à toute une classe d'algorithmes d'amincissement et de squelettisation; il y a, en effet, de multiples façons de définir un point terminal. Dans cette Note, nous donnons un seul exemple d'algorithme d'amincissement dérivé de ce schéma. Considérons la 6-adjacence pour X : avec cette relation d'adjacence, la structure 3D la plus élémentaire est un cube unité de $2 \times 2 \times 2$ points. Nous pouvons définir un point terminal comme étant un point qui n'appartient pas localement à une région 3D, c'est-à-dire qui n'appartient pas à un cube unité contenu dans X. Deux squelettes qui résultent de l'algorithme correspondant sont représentés à la figure 6.

Notons qu'il est facile, avec la notion de point *P*-simple, de mettre au point d'autres schémas d'amincissement parallèle.

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1. Introduction. – A simple point of an object is a point whose removal does not change the topology. Let us consider the figure 1 which consists in a two dimensional object in a square grid: the point a is not simple since its removal leads to disconnect the object; the point b is also not simple since its removal leads to merge two connected components of the complementary of X, *i.e.* to delete one hole in X; the points encircled are simple (when the so-called 4-adjacency is used, *see* [4], [6]). The notion of simple point is fundamental for all transformations where some topological features are to be preserved. Thinning algorithms are classical examples of such transformations: simple points of a 2D object are iteratively deleted until the object is reduced to a skeleton, *i.e.* until it consists only in curves.

A major problem which arises when designing thinning algorithms is that the simultaneous removal of simple points may change the topology of an object: for example, we see that, if we delete in parallel all simple points of the object depicted figure 1, it will be disconnected. A popular way for overcoming this problem is to consider a directional strategy for removing points in parallel: 2D points are classify in four types corresponding to the four directions α = North, South, East, Weast. A point of type α is a point of the object which has its immediate neighbor in the α direction which belongs to the complementary of the object. At each iteration, only simple points of a given type are considered for deletion. The four directions are alternatively used to that the thinning process is as symmetric as possible. This directional strategy has, in 2D, good topological properties: the topology of the object is preserved except that connected components composed of two points may be erased. When designing a thinning algorithm, it is therefore sufficient to check that these particular patterns are not deleted to have a sound algorithm.

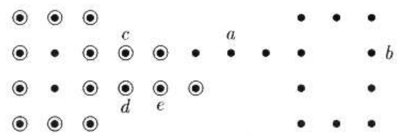
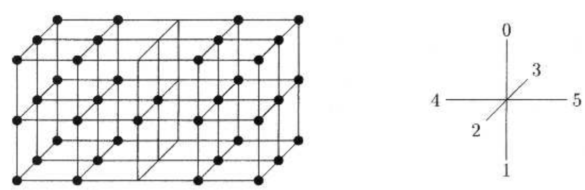


Fig. 1

Let us consider now the 3D case, for example let us consider an object in a cubic grid like the one depicted in figure 2 (a). For implementing a directional strategy, six directions α are now to be used [see *fig.* 2 (b)]. We see that, if all simple points of type 0 are removed, the object X will be disconnected. This shows that the classical directional strategy does not work in 3D. The purpose of this Note is to propose a strategy for removing points in parallel without altering the topology of 3D objects.



(a) (b)

Fig. 2

2. Basic notions. – A point $x \in \mathbf{Z}^3$ is defined by (x_1, x_2, x_3) with $x_i \in \mathbf{Z}$. We consider the three neighborhoods:

$$N_{26}(x) = \{x' \in \mathbf{Z}^3;\ \mathrm{Max}\,[|x_1 - x'_1|,\ |x_2 - x'_2|,\ |x_3 - x'_3|] \leqq 1\},$$
$$N_6(x) = \{x' \in \mathbf{Z}^3;\ |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leqq 1\},$$
$$N_{18}(x) = \{x' \in \mathbf{Z}^3;\ |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leqq 2\} \cap N_{26}(x).$$

We define $N_6^*(x) = N_6(x) \backslash \{x\}$, $N_{18}^*(x) = N_{18}(x) \backslash \{x\}$, $N_{26}^*(x) = N_{26}(x) \backslash \{x\}$.

Two points x and y are said to be *n-adjacent* ($n = 6, 18, 26$) if $y \in N_n^*(x)$. We call respectively 6, 18, 26-neighbors of x the points of $N_6^*(x)$, $N_{18}^*(x) \backslash N_6^*(x)$, $N_{26}^*(x) \backslash N_{18}^*(x)$: these points are represented figure 3 by full circles, full squares and circles, respectively.

An *n-path* is a sequence of points $x_0, \ldots, x_k$, with x_i n-adjacent to x_{i-1} for $i = 1 \ldots k$. If $x_0 = x_k$, the path is *closed.*

Let $X \subset \mathbf{Z}^3$. Two points $x \in X$ and $y \in X$ are *n-connected* if they are linked by an n-path included in X. The equivalence classes relative to this relation are the *n-connected components* of X. Let us consider the object X depicted in figure 4: X is included in a certain plane P. Let us suppose $n = 6$. We see that X is composed of four 6-connected components and X does not contain a non trivial closed 6-path; nevertheless the points of $\overline{X} \cap P$ are not 6-connected and the Jordan theorem does not hold. A paradox of the same kind arises if we consider the 26-adjacency for both X and $\overline{X}$. In order to have a correspondence between the topology of X and the one of $\overline{X}$, we have to consider two different kinds of adjacency for X and for $\overline{X}$ (*see* [4]): if we use an n-adjacency for X, we have to use another $\overline{n}$-adjacency for $\overline{X}$. In this Note, we consider that $(n, \overline{n}) = (26, 6)$ or $(6, 26)$.

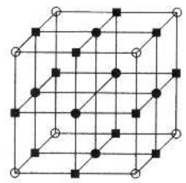


Fig. 3

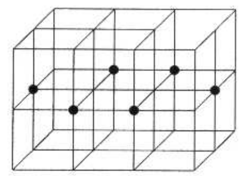


Fig. 4

The presence of an n-hole in X is detected whenever there is a closed n-path in X that cannot be deformed, in X, to a single point (*see* [3] and [1]). Let γ and γ' be two closed n-paths. We say that γ' is an *elementary deformation* of γ if γ' and γ are the same except in a little portion P of $\mathbf{Z}^3$; if $n = 26$, P is a $2 \times 2 \times 2$ unit cube and if $n = 6$, P is a 2×2 unit square. We say that γ' is a *deformation* of γ if there is a sequence of closed paths $\gamma_0, \ldots, \gamma_k$ such that $\gamma_0 = \gamma$, $\gamma_k = \gamma'$ and γ_i is an elementary deformation of γ_{i-1}. The relation of deformation induces some equivalence classes which correspond to the notion of holes. The use of a unit square and a unit cube for the 6- and the 26-adjacency, respectively, is necessary for the correspondence between the holes of X and of $\overline{X}$.

A point $x \in X$ is said to be *n-simple* if its removal does not change the topology in the sense that there is a one to one correspondence between the components, the holes of X and $\overline{X}$ and the components, the holes of $X \backslash \{x\}$ and $\overline{X} \cup \{x\}$, the n- and the $\overline{n}$-adjacency being used for X and $\overline{X}$, respectively.

If X is finite, the infinite connected component of $\overline{X}$ is the *background*, the other connected components of $\overline{X}$ are the *cavities*. We will also use the following notations:

– the set composed of all n-connected components of X is denoted $\mathcal{C}_n(X)$.

– the set of all n-connected components of X n-adjacent to a point x is denoted $\mathcal{C}_n^x(X)$.

Note that $\mathcal{C}_n(X)$ and $\mathcal{C}_n^x(X)$ are sets of subsets of X and not sets of points.

Finally $\#X$ stands for the cardinal of X.

3. *P*-SIMPLE POINTS. – Let us consider the two topological numbers (*see* [1], [5]):

DEFINITION 1. – Let $X \subset \mathbf{Z}^3$ and $x \in X$.

The *geodesic n-neighborhood of x inside X of order k* is the set $N_n^k(x, X)$ defined recursively by:

$$N_n^1(x, X) = N_n^*(x) \cap X$$

and

$$N_n^k(x, X) = \bigcup \{N_n(y) \cap N_{26}^*(x) \cap X,\ y \in N_n^{k-1}(x, X)\}.$$

Let

$$G_6(x, X) = N_6^2(x, X) \qquad \text{et} \qquad G_{26}(x, X) = N_{26}^1(x, X).$$

The *topological numbers* relative to X and x are the two numbers:

$$T_n(x, X) = \#\mathcal{C}_n[G_n(x, X)], \qquad \text{for} \quad n = 6, 26.$$

It can be seen that $T_6 = (x, X) = \#\mathcal{C}_6^x[N_{18}^*(x) \cap X]$ and $T_{26}(x, X) = \#\mathcal{C}_{26}^x[N_{26}^*(x) \cap X]$. These numbers lead to a very concise characterization of 3D simple points (*see* [1], [2]):

PROPOSITION 1. – *$x \in X$ is n-simple if and only if $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$.*

By using classical graph theoretic algorithms for searching connected components, the computations of $T_n(x, X)$ et $T_{\overline{n}}(x, \overline{X})$ can be done in a linear time complexity; *i.e.* they can be done respectively in $\mathcal{O}(k)$ and $\mathcal{O}(\overline{k})$ with $k = \#[G_n(x, X)]$ and $\overline{k} = \#[G_{\overline{n}}(x, \overline{X})]$.

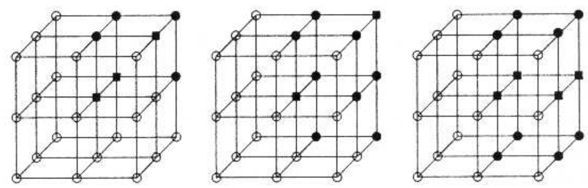
Let us introduce now the notion of P-simple point:

DEFINITION 2. – Let $X \subset \mathbf{Z}^3$.

Let $P \subset X$ and $x \in P$: x is *P_n-simple* if $\forall S \subset P \backslash \{x\}$, x is n-simple for $X \backslash S$.

Denote $S_n(P)$ the set of all P_n-simple points. A set D is *P_n-simple* if $D \subset S_n(P)$.

Some examples are given figure 5 : points belonging to $X \backslash P$, P and $\overline{X}$ are represented by full circles, full squares and circles, respectively. We see that, only the configuration (a) corresponds to a P_6-simple point and only the configurations (a) and (b) correspond to P_{26}-simple points.



(a) (b) (c)

Fig. 5

Let $P \subset X$ and $D \subset S_n(P)$. Let $D = \{x_1, ..., x_k\}$. Suppose we delete in sequence the points $x_1, ..., x_k$. At the step i of this procedure, since x_i is P-simple, x_i is simple for the set $X \backslash D_i$ with $D_i = \{x_1, ..., x_{i-1}\}$ and we do not change the topology by deleting x_i. Hence X and X/D are topologically equivalent in the sense described above: any algorithm which removes only P-simple sets is guaranteed to keep the topology unchanged.

Since the characterization of a simple point is local, we can derive a local characterization of P-simple points; since $G_n(x, X) \subset N^*_{26}(x)$ and $G_{\overline{n}}(x, \overline{X}) \subset N^*_{26}(x)$, we have:

$$x \text{ is } P_n\text{-simple} \quad \Leftrightarrow \quad \forall S \subset P \cap N^*_{26}(x), \qquad x \text{ is } n\text{-simple for } X \backslash S.$$

Thus, the checking for P-simple points could be made by checking the condition $[T_n(x, X \backslash S) = 1 \text{ and } T_{\overline{n}}(x, \overline{X \backslash S}) = 1]$ for all sets $S \subset P \cap N^*_{26}(x)$. We see that this method leads to a time complexity (1) which is worst than $\mathcal{O}(2^k)$: this method is useless since too computationaly expensive. The following proposition allows to have a very efficient way for checking P-simple points:

PROPOSITION 2. – *Let $P \subset X$ and $x \in P$; we denote $R = X \backslash P$:*

$$x \text{ is } P_n\text{-simple} \quad \Leftrightarrow \quad \begin{cases} 1)\, T_n(x, R) = 1 \\ 2)\, T_{\overline{n}}(x, \overline{X}) = 1 \\ 3)\, \forall y \in N^*_n(x) \cap P,\ N^*_n(y) \cap G_n(x, R) \neq \varnothing \\ 4)\, \forall y \in N^*_{\overline{n}}(x) \cap P,\ N^*_{\overline{n}}(y) \cap G_{\overline{n}}(x, \overline{X}) \neq \varnothing. \end{cases}$$

Proof. – (i) Suppose x is P_n-simple. Then x is n-simple for X and for $R \cup \{x\}$, hence $T_{\overline{n}}(x, \overline{X}) = 1$ and $T_n(x, R \cup \{x\}) = T_n(x, R) = 1$.

Suppose $\exists\, y \in N^*_n(x) \cap P$ such that $N^*_n(y) \cap G_n(x, R) = \varnothing$. It means that if we delete all points of P except y and x, we will have two connected components n-adjacent to x: one component composed of points of R (since $T_n(x, R) = 1$) and one component composed of the point y since y is n-adjacent to x and y is not n-adjacent to a point of $G_n(x, R)$. Hence $\#\mathcal{C}_n(G_n(x, (X\backslash P) \cup \{x, y\})) = 2$, x would not be simple for the set $(X\backslash P) \cup \{x, y\}$ and therefore it would not be *P*-simple.

Suppose $\exists\, y \in N^*_{\overline{n}}(x) \cap P$ such that $N^*_{\overline{n}}(y) \cap G_{\overline{n}}(x, \overline{X}) = \varnothing$. Suppose we delete y. We have two connected components of $\overline{X} \cup \{y\}$ $\overline{n}$-adjacent to x: one component composed of points of $\overline{X}$ [since $T_{\overline{n}}(x, \overline{X}) = 1$] and one component composed of the point y since y is $\overline{n}$-adjacent to x and y is not $\overline{n}$-adjacent to a point of $G_{\overline{n}}(x, \overline{X})$. Hence $\#\mathcal{C}_{\overline{n}}(G_{\overline{n}}(x, \overline{X} \cup \{y\})) = 2$, x would not be simple for the set $X\backslash\{y\}$ and therefore it would not be *P*-simple.

(ii) Suppose the four conditions are fulfilled. Since all points of *P* adjacent to x are adjacent to some points of $G_n(x, R)$ (condition 3), points of *P* cannot generate a component of *X* adjacent to x; so we have $T_n(x, X) = \#\mathcal{C}_n(G_n(x, X)) = \#\mathcal{C}_n(G_n(x, R)) = T_n(x, R) = 1$ (condition 1). Since $T_{\overline{n}}(x, \overline{X}) = 1$ (condition 2), x is a simple point for *X*.

Let s be a point of $P \cap N^*_{26}(x)$. Suppose we delete s. We see that:

1) The condition $T_n(x, R) = 1$ is always true.

2) Since the condition 4), the deletion of s cannot gnerate a new component adjacent to x in $\overline{X}$; so we have $\#\mathcal{C}_{\overline{n}}(G_{\overline{n}}(x, \overline{X} \cup \{s\})) = \#\mathcal{C}_{\overline{n}}(G_{\overline{n}}(x, \overline{X})) = 1$ (condition 2). Hence $T_{\overline{n}}(x, \overline{X} \cup \{s\}) = 1$.

3) The remaining points of *P* still fulfill the condition 3):

$$\forall\, y \in N^*_n(x) \cap (P\backslash\{s\}), \quad N^*_n(y) \cap G_n(x, R) \neq \varnothing.$$

4) These points also fulfill the condition 4):

$$\forall\, y \in N^*_{\overline{n}}(x) \cap (P\backslash\{s\}), \quad N^*_{\overline{n}}(y) \cap G_{\overline{n}}(x, \overline{X} \cup \{s\}) \neq \varnothing.$$

So x still fulfills the four conditions; therefore, as seen above, it means that x is a simple point for $X\backslash\{s\}$. By induction, we see that we can successively delete the points of a set $S \subset P \cap N^*_{26}(x)$: at each step of this procedure x is simple since it fulfills the four conditions. Hence x is simple for $X\backslash S$: since the above local characterization, x is a P_n-simple point. □

The conditions 1) and 3) may be implemented in $\mathcal{O}(k)$ and the conditions 2) and 4) in $\mathcal{O}(\overline{k})$: the characterization of *P*-simple points may be done in linear time.

4. A Parallel thinning scheme. – As an example of application of the notion of *P*-simple point, we propose a new 3D parallel thinning scheme which is a sound directional thinning scheme. We denote $d^\alpha(x)$, $\alpha = 0, ..., 5$, the 6-neighbors of x [see *fig.* 2 (*b*)]. Let $d^\alpha(X) = \{x \in X,\ d^\alpha(x) \in \overline{X}\}$. The thinning scheme consists in successively considering the 6 directions α and to keep as candidates for deletion only the points belonging to $d^\alpha(X)$. Furthermore an end point characterization is used in order to prevent from deletion surfaces or curves when they appear during the thinning process. The set of end points of a set *Y* is denoted $E(Y)$. The thinning scheme may be described as follows, we denote $X^0 = X$, [ ] stands for the modulo function:

$$X^{i+1} = X^i\backslash S_n(P^i), \qquad \text{with} \quad P^i = \overline{E(X^i)} \cap d^{i\,[6]}(X^i).$$

The *skeleton* of *X* is the set $SK(X) = X^k$, k being such that $X^k = X^{k+6}$.

This thinning scheme leads to a broad class of thinning algorithms since there are many ways for defining end points. In this Note, we just give one example of a thinning algorithm derived from this scheme. Let us use the 6-adjacency for *X*: when using this adjacency, the most elementary 3D pattern is a unit cube, *i.e.* a $2 \times 2 \times 2$ cube. We may define an end point as a point which does not locally belongs to a 3D region, *i.e.* which does not belong to a unit cube included in X. With this definition, the thinning scheme leads to a thinning algorithm the results of which are illustrated figure 6. It could be seen that, if all the possible sets X are contained in a $l \times l \times l$ cube, the time complexity of this algorithm is in $\mathcal{O}(l^4)$; with l^3 processors, it may be reduced to $\mathcal{O}(l)$.

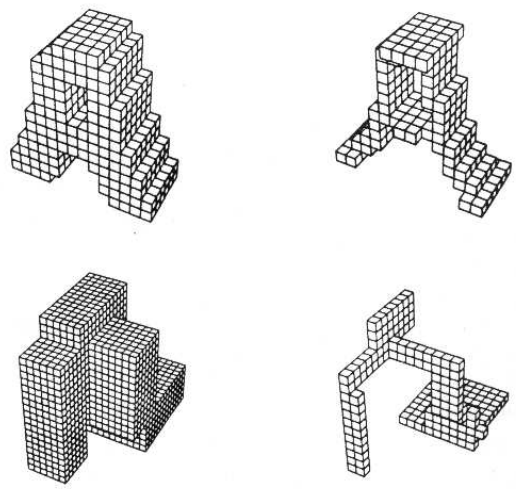


Fig. 6

Note that, with the notion of P-simple point, it is very easy to design other thinning schemes.

([1]) The number k is bounded by 26 since $G_n(x, X) \subset N_{26}(x)$.

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